A New Tripartite Coherent-Entangled State Generated by An Asymmetric Beam-Splitter and Its Applications

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Abstract. A new kind of tripartite coherent-entangled state (CES) $|\beta, \gamma, x\rangle_{\mu\nu\tau}$ is proposed, which exhibits the properties of both coherence and entanglement. We investigate its completeness and orthogonality, and find it can make up a representation of tripartite CES. A protocol for generating the tripartite CES is proposed using asymmetric beam splitter. Applications of the tripartite CES in quantum optics are also presented.

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1. Introduction

The concept of entanglement, originated by Einstein, Podolsky and Rosen (EPR) for arguing the incompleteness of quantum mechanics, plays a key role in understanding some fundamental problems in quantum mechanics. Quantum entangled states have been vastly studied by physicists due to their potential usage in quantum information and quantum communication. In a quantum entangled system, a measurement performed on one part of the system provides information about the remaining part, and this is now known as the basic feature of quantum mechanics, weird though it seems. For a good understanding entanglement, Ref. [1] will be useful. Beyond all entangled states, the continuous variable entangled states are of great application in quantum optics and atomics area, where the continuous variables are just the quadrature phase of optics field. Detail acquaintance of continuous variable can refer to Ref. [2, 3]. And it can be inferred that continuous variable entanglement states such as topological entanglement may play an important roll in understanding famous and obscure phenomenons in low temperature physics such as the fractional electron charge effect [4]. On the other hand, the theoretical research has go ahead of experiment to construct various continuous variable entangled states: idealized EPR state $|\eta\rangle$ [5], two mode coherent entangled state $|\alpha, x\rangle$ [6], and arbitrary multi-mode entangled state [7] and so on. All these mentioned states have been constructed and property-analyzed basing on IWOP technique [8–11]. Contrast to classical quantum optical states, these states present nonclassical properties such as partial non-positive of Wigner distributive and Mandel factor [12], divergence in special point in phase space of Glauber-Sudarshan representation [13]. Among these states, CES is of special interesting due to its intrinsic nature of the merging of coherence and entanglement. As far as we know, multipartite CES have been write down in Ref. [6, 15]. Here we propose a new tripartite CES $|\beta, \gamma, x\rangle_{\mu\nu\tau}$, which is the generalization version of the old tripartite CES $|\beta, \gamma, x\rangle$. This generalization is not trivial. As tripartite CES, we find that it can play as the continuous base in Hilbert space of square-integrated property, after checking its completeness and orthogonality.

This paper is arranged as follows, in Sec. 2 the explicit form of tripartite CES $|\beta, \gamma, x\rangle_{\mu\nu\tau}$ is present in Fock space by virtue of the technique of integration within an ordered product of operators (IWOP), and then some main properties are analyzed in Sec. 3. The protocol for generating the tripartite CES is proposed In Sec. 4 using asymmetric beam splitter. Sec. 5 is devoted to briefly discussing some potential applications of $|\beta, \gamma, x\rangle_{\mu\nu\tau}$ in quantum optics. A brief conclusion is presented in the last section.

2. The Introduction of tripartite CES

For a physical state, it is hoped that it can span a complete space. For example, the Fock state and the coherent state are both complete. It has been shown [7] that by constructing miscellaneous normally ordered Gaussian integration operators, which are unity operators, and then considering their decomposition of unity we may derive new quantum mechanical states possessing the completeness relation and orthogonality. For example, from the normally ordered Gaussian form of unity

$$\int \frac{\mathrm{d}^2 z}{\pi} : \exp\left[-(z^* - a^\dagger)(z - a)\right] := 1 \tag{1}$$

and using the normal ordering form of vacuum state projector $|0\rangle\langle 0| =: \exp(-a^\dagger a):$ [16, 17], where : : is signal of normal ordering, we can make the decomposition

$$: \exp\left[-(z^* - a^\dagger)(z - a)\right] := |z\rangle\langle z| \tag{2}$$

so the form of coherent state $|z\rangle = \exp(-\frac{|z|^2}{2} + za^\dagger)|0\rangle$ emerges. Similarly, by examining

$$\int \frac{\mathrm{d}^2\eta}{\pi} : \exp\left[-(\eta^* - a_1^\dagger + a_2)(\eta - a_1 + a_2^\dagger)\right] := 1 \tag{3}$$

and decomposing the integrand in Equation (3) we observe the emergence of bi-particle ideal EPR state $|\eta\rangle = \exp[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger]|00\rangle$ [5]. And if we go further, by decomposition the following unity of the Gaussian operator integration within normal ordering

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{2\pi} : \exp\left[-\left(x - \frac{\mu X_1 + \nu X_2}{\sqrt{2}\lambda}\right)^2\right] \\ &\times \exp\left\{-\frac{1}{2}\left[\alpha^* - \frac{1}{\lambda}(\nu a_1^\dagger - \mu a_2^\dagger)\right]\left[\alpha - \frac{1}{\lambda}(\nu a_1 - \mu a_2)\right]\right\} := 1\end{aligned} \tag{4}$$

after the decomposition, we can get the expression of the state

$$\begin{aligned}|\alpha, x\rangle_{\mu\nu} = \exp&\left[-\frac{1}{2}x^2 - \frac{1}{4}|\nu\alpha|^2 + \lambda\alpha a_1^\dagger + \frac{\mu}{\lambda}\left(x - \frac{\alpha\mu}{2}\right)a_1^\dagger\right. \\ &\left.+\frac{\nu}{\lambda}\left(x - \frac{\alpha\mu}{2}\right)a_2^\dagger - \frac{1}{(2\lambda)^2}\left(\mu a_1^\dagger + \nu a_2^\dagger\right)^2\right]|00\rangle\end{aligned} \tag{5}$$

This is the new bipartite CES proposed in Ref. [18], with $\mu^2 + \nu^2 = 2\lambda^2$. Using the bosonic communicative relation $[a_i, a_j^\dagger] = \delta_{ij}$, we have

$$a_1|\alpha, x\rangle_{\mu\nu} = \left[\alpha\lambda + \frac{\mu}{\lambda}\left(x - \frac{\alpha\mu}{2}\right) - \frac{\mu}{2\lambda^2}(\mu a_1^\dagger + \nu a_2^\dagger)\right]|\alpha, x\rangle_{\mu\nu} \tag{6a}$$

$$a_2|\alpha, x\rangle_{\mu\nu} = \left[\frac{\nu}{\lambda}\left(x - \frac{\alpha\mu}{2}\right) - \frac{\nu}{2\lambda^2}(\mu a_1^\dagger + \nu a_2^\dagger)\right]|\alpha, x\rangle_{\mu\nu} \tag{6b}$$

which satisfy the following eigenequations

$$\frac{1}{2}(\mu X_1 + \nu X_2)|\alpha, x\rangle_{\mu\nu} = \frac{\lambda x}{\sqrt{2}}|\alpha, x\rangle_{\mu\nu} \tag{7a}$$

$$(\nu a_1 - \mu a_2)|\alpha, x\rangle_{\mu\nu} = \nu\alpha\lambda|\alpha, x\rangle_{\mu\nu}, \tag{7b}$$

which means $|\alpha, x\rangle_{\mu\nu}$ is actually the common eigenvector of $(\mu X_1 + \nu X_2)$ and $(\nu a_1 - \mu a_2)$, and $[(\mu X_1 + \nu X_2), (\nu a_1 - \mu a_2)] = 0$, and

$$X_i = (a_i + a_i^\dagger)/\sqrt{2}, \qquad P_i = (a_i - a_i^\dagger)/(\sqrt{2}\mathrm{i}) \tag{8}$$

However, if we want to obtain the expression of tripartite CES, it may become tedious to construct such a complex quadratic gaussian polynomial of three-mode of generate operator in entangled form to derive tripartite CES. Fortunately, we can stride over this problem just oppositely, first constructing the tripartite counterpart formally analogue to bipartite CES, then checking it satisfies the similar relationship Equation (3). Along this way, tripartite CES $|\beta, \gamma, x\rangle_{\mu\nu\tau}$ can be introduced as

$$|\beta, \gamma, x\rangle_{\mu\nu\tau} = \exp\left\{\left[-\frac{3}{4}x^2 - \frac{1}{6\nu}(\beta^*\gamma + \beta\gamma^*)\mu\tau^2 - \frac{1}{6}|\gamma|^2\tau^2\left(1 + \frac{\mu^2}{\nu^2}\right)\right.\right.$$

$$- \frac{1}{6}|\beta|^2\left(\nu^2+\tau^2\right)\Big] + \frac{1}{3\lambda}\Bigg[\left(\beta(\nu^2+\tau^2)+\frac{\gamma\mu\tau^2}{\nu}+3x\mu\right)a_1^\dagger + \left(-\beta\mu\nu+\gamma\tau^2+3x\nu\right)a_2^\dagger + \left(-\frac{\gamma(\mu^2+\nu^2)\tau}{\nu}-\beta\mu\tau+3x\tau\right)a_3^\dagger\Bigg] - \frac{1}{6\lambda^2}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)^2\Bigg\}|000\rangle \quad (9)$$

where μ, ν, τ are three independent parameters, and $3\lambda^2 = \mu^2+\nu^2+\tau^2$, this identity hold to make sure it can be generated by beam-splitter which will be instructed in Sec. 4. In particular, when $\mu=\nu=\tau=1$, Equation (9) will reduce to

$$|\beta,\gamma,x\rangle = \exp\Bigg\{-\frac{3}{4}x^2 - \frac{1}{6}\left(\beta\gamma^*+\beta^*\gamma+2|\beta|^2+2|\gamma|^2\right) + \left[x+\frac{1}{3}(2\beta+\gamma)\right]a_1^\dagger + \left[x+\frac{1}{3}(-\beta+\gamma)\right]a_2^\dagger + \left[x+\frac{1}{3}(-\beta-2\gamma)\right]a_3^\dagger - \frac{1}{6}(a_1^\dagger+a_2^\dagger+a_3^\dagger)^2\Bigg\}|000\rangle \quad (10)$$

This is the so called tripartite CES introduced in Ref. [14]. Using the bosonic commutative relation $[a_i, a_j^\dagger] = \delta_{ij}$, we have

$$a_1|\beta,\gamma,x\rangle_{\mu\nu\tau} = \frac{1}{3\lambda}\left[\left(\beta(\nu^2+\tau^2)+\frac{\gamma\mu\tau^2}{\nu}+3x\mu\right) - \frac{\mu}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (11a)$$

$$a_2|\beta,\gamma,x\rangle_{\mu\nu\tau} = \frac{1}{3\lambda}\left[\left(-\beta\mu\nu+\gamma\tau^2+3x\nu\right) - \frac{\nu}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (11b)$$

$$a_3|\beta,\gamma,x\rangle_{\mu\nu\tau} = \frac{1}{3\lambda}\left[\left(-\frac{\gamma(\mu^2+\nu^2)\tau}{\nu}-\beta\mu\tau+3x\tau\right) - \frac{\tau}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (11c)$$

Combining the equations (10), we obtain the eigenequations of tripartite CES $|\beta,\gamma,x\rangle_{\mu\nu\tau}$

$$\frac{1}{3}(\mu X_1+\nu X_2+\tau X_3)|\beta,\gamma,x\rangle_{\mu\nu\tau} = \frac{\lambda x}{\sqrt{2}}|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (12a)$$

$$(\nu a_1-\mu a_2)|\beta,\gamma,x\rangle_{\mu\nu\tau} = \nu\beta\lambda|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (12b)$$

$$(\tau a_2-\nu a_3)|\beta,\gamma,x\rangle_{\mu\nu\tau} = \tau\gamma\lambda|\beta,\gamma,x\rangle_{\mu\nu\tau} \quad (12c)$$

So we see that $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ is actually the common eigenvector of $\frac{1}{3}(\mu X_1 + \nu X_2+\tau X_3)$, $(\nu a_1-\mu a_2)$ and $(\tau a_2-\nu a_3)$, and $[(\mu X_1+\nu X_2+\tau X_3),(\nu a_1-\mu a_2)] = [(\mu X_1+\nu X_2+\tau X_3),(\tau a_2-\nu a_3)] = [(\nu a_1-\mu a_2),(\tau a_2-\nu a_3)] = 0$.

3. Main Properties Of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$

In Sec. 2, we construct the tripartite CES $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ just oppositely to traditional ways, and now we will check its orthogonality and completeness, to prove it span the Hilbert space of tripartite states, and so make up a new kind of representation.

3.1. Orthogonal Property

We investigate whether $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ is mutual orthogonal or not. Explicitly, using the eigenequations of tripartite CES, we examine the following matrix elements:

$$ {}_{\mu\nu\tau}\langle\beta',\gamma',x'|\,\frac{\mu X_1+\nu X_2+\tau X_3}{3}\,|\beta,\gamma,x\rangle_{\mu\nu\tau} = \frac{\lambda x'}{\sqrt{2}}\;{}_{\mu\nu\tau}\langle\beta',\gamma',x'|\beta,\gamma,x\rangle_{\mu\nu\tau} \qquad (13a)$$

$$ = \frac{\lambda x}{\sqrt{2}}\;{}_{\mu\nu\tau}\langle\beta',\gamma',x'|\beta,\gamma,x\rangle_{\mu\nu\tau} \qquad (13b)$$

which leads to

$$ {}_{\mu\nu\tau}\langle\beta',\gamma',x'|\beta,\gamma,x\rangle_{\mu\nu\tau}\,(x'-x)=0 \qquad (14)$$

To derive the exact express of ${}_{\mu\nu\tau}\langle\beta',\gamma',x'|\beta,\gamma,x\rangle_{\mu\nu\tau}$, we will use the over-completeness relation of the three-mode coherent state

$$ \int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2\mathrm{d}^2z_3}{\pi^3}\,|z_1,z_2,z_3\rangle\langle z_1,z_2,z_3|=1 \qquad (15)$$

where

$$ \begin{aligned} |z_1,z_2,z_3\rangle &= D_1(z_1)D_2(z_2)D_3(z_3)\,|000\rangle \\ &= \exp\left[-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_3|^2\right)+z_1a_1^\dagger+z_2a_2^\dagger+z_3a_3^\dagger\right]|000\rangle \end{aligned} \qquad (16)$$

and $D_i(z)=\exp(za_i^\dagger-z^*a_i)$. Using the definition of Tripartite CES in Equation (10), the overlap is

$$ \begin{aligned} &\langle z_1,z_2,z_3|\beta,\gamma,x\rangle_{\mu\nu\tau} \\ &=\exp\left\{\left[-\frac{3}{4}x^2-\frac{1}{6\nu}(\beta^*\gamma+\beta\gamma^*)\mu\tau^2-\frac{1}{6}|\gamma|^2\tau^2\left(1+\frac{\mu^2}{\nu^2}\right)-\frac{1}{6}|\beta|^2\left(\nu^2+\tau^2\right)\right]\right. \\ &\quad+\frac{1}{3\lambda}\left[\left(\beta(\mu^2+\tau^2)+\frac{\gamma\mu\tau^2}{\nu}+3x\mu\right)z_1^*+\left(-\beta\mu\nu+\gamma\tau^2+3x\nu\right)z_2^*\right. \\ &\quad\left.+\left(-\frac{\gamma(\mu^2+\nu^2)\tau}{\nu}-\beta\mu\tau+3x\tau\right)z_3^*\right] \\ &\quad\left.-\frac{1}{6\lambda^2}\left(\mu z_1^*+\nu z_2^*+\tau z_3^*\right)^2-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_3|^2\right)\right\} \end{aligned} \qquad (17)$$

To calculate ${}_{\mu\nu\tau}\langle\beta',\gamma',x'|\beta,\gamma,x\rangle_{\mu\nu\tau}$

$$ \begin{aligned} \langle\beta',\gamma',x'|\beta,\gamma,x\rangle &= \int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2\mathrm{d}^2z_3}{\pi^3}\,\langle\beta',\gamma',x'|z_1,z_2,z_3\rangle\,\langle z_1,z_2,z_3|\beta,\gamma,x\rangle \\ &= \exp\left\{-\frac{\mu^2+\nu^2}{6\nu^2}\left[\nu^2(|\beta|^2+|\beta'|^2)+\tau^2\left(|\gamma|^2+|\gamma'|^2\right)\right]\right. \\ &\qquad-\frac{\mu}{6\nu}\tau^2\left[\beta\gamma^*+\beta^*\gamma+\beta'\gamma'^*+\beta'^*\gamma'-2(\beta\gamma'^*+\beta'^*\gamma)\right] \\ &\qquad\left.+\frac{\nu^2+\tau^2}{3\nu^2}(\nu^2\beta\beta'^*+\mu^2\gamma\gamma'^*)\right\}\delta(x-x') \end{aligned} \qquad (18)$$

In deriving Equation (18), we have used the mathematical formula

$$ \int\frac{\mathrm{d}^2z}{\pi}\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\left[\frac{-\zeta\xi\eta+\xi^2g+\eta^2f}{\zeta^2-4fg}\right] \qquad (19)$$

with its convergent condition

$$\mathrm{Re}(\xi + f + g) < 0, \quad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\xi + f + g}\right) < 0$$

or

$$\mathrm{Re}(\xi - f - g) < 0, \quad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\xi - f - g}\right) < 0$$

and the limiting form of Dirac's delta function

$$\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{\varepsilon}\right) \tag{20}$$

3.2. Completeness Relation

Now we shall check whether $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ possesses the completeness relation. By virtue of the technique of IWOP, and the normal ordered product of the three-mode vacuum projector

$$|000\rangle\langle 000| =: \exp(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3) : \tag{21}$$

we can smoothly prove the completeness relation of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$

$$\begin{aligned}
&\int \frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{6\pi}}\,|\beta,\gamma,x\rangle_{\mu\nu\tau}\,{}_{\mu\nu\tau}\langle\beta,\gamma,x| \\
&= \int \frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{6\pi}} : \exp\left\{-\frac{1}{3}\left(\frac{3}{\sqrt{2}}x - \frac{\mu X_1 + \nu X_2 + \tau X_3}{\lambda}\right)^2\right. \\
&\qquad -\frac{1}{3}\left[\left(\nu\beta^* - \frac{\nu a_1^\dagger - \mu a_2^\dagger}{\lambda}\right)\left(\nu\beta - \frac{\nu a_1 - \mu a_2}{\lambda}\right)\right] \\
&\qquad -\frac{1}{3}\left[\left(\tau\gamma^* - \frac{\tau a_2^\dagger - \nu a_3^\dagger}{\lambda}\right)\left(\tau\gamma - \frac{\tau a_2 - \nu a_3}{\lambda}\right)\right] \\
&\qquad \left.-\frac{1}{3}\left[\left(\frac{\tau}{\nu}(\nu\beta^* + \mu\gamma^*) - \frac{\tau a_1^\dagger - \mu a_3^\dagger}{\lambda}\right)\left(\frac{\tau}{\nu}(\nu\beta + \mu\gamma) - \frac{\tau a_1 - \mu a_3}{\lambda}\right)\right]\right\} : \\
&= \frac{3}{\tau^2\lambda^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{6\pi}} : \exp\left[-\frac{1}{3}\left(\frac{3}{\sqrt{2}}x - \frac{\mu X_1 + \nu X_2 + \tau X_3}{\lambda}\right)^2\right] : \qquad (22) \\
&= \frac{1}{\tau^2\lambda^2} \qquad (23)
\end{aligned}$$

and also have

$$\begin{aligned}
&\int \frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}\,|\beta,\gamma,x\rangle_{\mu\nu\tau}\,{}_{\mu\nu\tau}\langle\beta,\gamma,x| = \\
&\frac{3}{\tau^2\lambda^2} : \exp\left[-3\left(\frac{1}{\sqrt{2}}x - \frac{\mu X_1 + \nu X_2 + \tau X_3}{3\lambda}\right)^2\right] :
\end{aligned} \tag{24}$$

3.3. The Conjugate State of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$

According to communication relationship between mechanic operator and quantum state, once we known tripartite CES, we can derive its conjugate state. Three-particle's total momentum is $P=\Sigma_{i=1}^{3}P_i$, $(P_i=(a_i-a_i^\dagger)/(\mathrm{i}\sqrt{2}))$, P $(\nu a_1-\mu a_2)$, and $(\tau a_2-\nu a_3)$ are permutable with each other as well, we make great efforts to find their common eigenvector with eigenvalues $\lambda p/\sqrt{2}$, $\nu\sigma\lambda$ and $\tau\kappa\lambda$, expressed as $|\sigma,\kappa,p\rangle_{\mu\nu\tau}$:

$$\begin{aligned}|\sigma,\kappa,p\rangle_{\mu\nu\tau} = \exp\Bigg\{&\left[-\frac{3}{4}p^2-\frac{1}{6\nu}(\sigma^*\kappa+\sigma\kappa^*)\mu\tau^2-\frac{1}{6}|\kappa|^2\tau^2\left(1+\frac{\mu^2}{\nu^2}\right)-\frac{1}{6}|\sigma|^2\left(\nu^2+\tau^2\right)\right]\\ &+\frac{1}{3\lambda}\left[\left(\sigma(\nu^2+\tau^2)+\frac{\kappa\mu\tau^2}{\nu}+3\mathrm{i}p\mu\right)a_1^\dagger+\left(-\sigma\mu\nu+\kappa\tau^2+3\mathrm{i}p\nu\right)a_2^\dagger\right.\\ &+\left.\left(-\frac{\kappa(\mu^2+\nu^2)\tau}{\nu}-\sigma\mu\tau+3\mathrm{i}p\tau\right)a_3^\dagger\right]\\ &+\frac{1}{6\lambda^2}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)^2\Bigg\}|000\rangle \end{aligned} \tag{25}$$

The results after annihilation operators acting on $|\sigma,\kappa,p\rangle_{\mu\nu\tau}$ respectively are

$$\begin{aligned} a_1|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \frac{1}{3\lambda}\left[\left(\sigma(\nu^2+\tau^2)+\frac{\kappa\mu\tau^2}{\nu}+3\mathrm{i}p\mu\right)\right.\\ &\qquad\left.+\frac{\mu}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\sigma,\kappa,p\rangle_{\mu\nu\tau} \qquad (26a)\\ a_2|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \frac{1}{3\lambda}\left[(-\sigma\mu\nu+\kappa\tau^2+3\mathrm{i}p\nu)\right.\\ &\qquad\left.+\frac{\nu}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\sigma,\kappa,x\rangle_{\mu\nu\tau} \qquad (26b)\\ a_3|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \frac{1}{3\lambda}\left[\left(-\frac{\kappa(\mu^2+\nu^2)\tau}{\nu}-\sigma\mu\tau+3\mathrm{i}p\tau\right)\right.\\ &\qquad\left.+\frac{\tau}{\lambda}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)\right]|\sigma,\kappa,p\rangle_{\mu\nu\tau} \qquad (26c)\end{aligned}$$

from the above equations, we get similar expressions of its eigenequations as those of tripartite CES

$$\begin{aligned}\frac{1}{3}(\mu P_1+\nu P_2+\tau P_3)|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \frac{\lambda p}{\sqrt{2}}|\beta,\gamma,x\rangle_{\mu\nu\tau} \qquad (27a)\\ (\nu a_1-\mu a_2)|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \nu\sigma\lambda|\sigma,\kappa,p\rangle_{\mu\nu\tau} \qquad (27b)\\ (\tau a_2-\nu a_3)|\sigma,\kappa,p\rangle_{\mu\nu\tau} &= \tau\kappa\lambda|\sigma,\kappa,p\rangle_{\mu\nu\tau} \qquad (27c)\end{aligned}$$

So far we get tripartite momentum CES. Using the IWOP technique we can prove

$$\begin{aligned}&\int\frac{\mathrm{d}^2\sigma}{\pi}\frac{\mathrm{d}^2\kappa}{\pi}|\sigma,\kappa,p\rangle_{\mu\nu\tau}\,{}_{\mu\nu\tau}\langle\sigma,\kappa,p| =\\ &\frac{3}{\tau^2\lambda^2}:\exp\left[-3\left(\frac{1}{\sqrt{2}}p-\frac{\mu P_1+\nu P_2+\tau P_3}{3\lambda}\right)^2\right]:\end{aligned} \tag{28}$$

so the completeness integration also holds, i.e.,

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{\sqrt{6\pi}}\int\frac{\mathrm{d}^2\sigma}{\pi}\frac{\mathrm{d}^2\kappa}{\pi}|\sigma,\kappa,p\rangle_{\mu\nu\tau}\,{}_{\mu\nu\tau}\langle\sigma,\kappa,p|=\frac{1}{\tau^2\lambda^2} \tag{29}$$

Thus $|\sigma,\kappa,p\rangle_{\mu\nu\tau\ \mu\nu\tau}$ is the conjugate state of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$.

4. Generating Tripartite CES By Asymmetric Beam-splitter(BS)

The tripartite CES as we show upper, can be generated by asymmetric BS operator. One function of BS it to generate entangled state [19], and operator representation of BS operating on incident optic field can be expressed (with phase-free) by [20]

$$B_{ij}(\theta) = \exp\left[-\theta\left(a_i^\dagger a_j - a_i a_j^\dagger\right)\right] \tag{30}$$

Letting the ideal single-mode maximal-squeezed state in mode 1, expressed by $|x=0\rangle_1 = exp\left[-\frac{1}{2}a_1^\dagger\right]|0\rangle_1$, and the vacuum state $|0\rangle_{2,3}$ in mode 2, 3 respectively enter the two input ports of two sequential asymmetric BS and get overlapped, we have

$$B_{23}(\varphi)B_{12}(\theta)\exp\left[-\frac{1}{2}a_1^\dagger\right]|0\rangle_1\otimes|0\rangle_2\otimes|0\rangle_3 = \exp\left[-\frac{1}{2}\left(a_1^\dagger\cos(\theta)+a_2^\dagger\sin(\theta)\cos(\varphi)+a_3^\dagger\sin(\theta)\sin(\varphi)\right)^2\right]|000\rangle \tag{31}$$

Since $\cos^2(\theta)+\sin^2(\theta)\cos^2(\varphi)+\sin^2(\theta)\sin^2(\varphi)=1$, so λ is introduced as $3\lambda^2 = \mu^2+\nu^2+\tau^2$ as illustrated in Sec. 2. When $\theta = \arccos(\frac{\mu}{\sqrt{3}\lambda})$ and $\varphi = \arccos(\frac{\nu}{\sqrt{\nu^2+\tau^2}})$, the state out of the two sequential BS in Equation (31) becomes

$$\exp\left[-\frac{1}{6\lambda^2}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)^2\right]|000\rangle_{123} \tag{32}$$

which is a three-mode squeezed state. Then operating three sequential displacement operators $D_1(\epsilon_1)$, $D_2(\epsilon_2)$, $D_3(\epsilon_3)$ on three individual mode, where $D_i(\epsilon)$ writes

$$D_i(\epsilon) = \exp(\epsilon a_i^\dagger - \epsilon^* a_i) \tag{33}$$

and the displacements ϵ_1, ϵ_2, ϵ_3 are

$$\epsilon_1 = \frac{2\beta\nu(\nu^2+\tau^2)+2\gamma\mu\tau^2+3x\mu\nu}{6\nu\lambda} \tag{34a}$$

$$\epsilon_2 = \frac{-2\beta\mu\nu+2\gamma\tau^2+3x\nu}{6\lambda} \tag{34b}$$

$$\epsilon_3 = \frac{-2\beta\mu\nu\tau-2\gamma\tau(\mu^2+\nu^2)+3x\nu\tau}{6\nu\lambda} \tag{34c}$$

After these three sequential displacements, the ideal three-mode asymmetry squeezed state will becomes

$$\begin{aligned} D_1(\epsilon_1)D_2(\epsilon_2)D_3(\epsilon_3)&\exp\left[-\frac{1}{6\lambda^2}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)^2\right]|000\rangle \\ =\ &\exp\left\{-\frac{\epsilon_1\epsilon_1^*+\epsilon_2\epsilon_2^*+\epsilon_3\epsilon_3^*}{2}+\epsilon_1 a_1^\dagger+\epsilon_2 a_2^\dagger+\epsilon_3 a_3^\dagger\right. \\ &\left.-\frac{1}{6\lambda^2}\left(\mu(a_1^\dagger-\epsilon_1^*)+\nu(a_2^\dagger-\epsilon_2^*)+\tau(a_3^\dagger-\epsilon_3^*)\right)^2\right\}|000\rangle \end{aligned} \tag{35}$$

Substitute the expression of ϵ_i into Equation (35), we find the state is tripartite CES compared with Equation (10). And experimentally, we can achieve these displacements (eg. $D_1(\epsilon_1)$), by reflecting the light field $\exp\left[-\frac{1}{6\lambda^2}\left(\mu a_1^\dagger+\nu a_2^\dagger+\tau a_3^\dagger\right)^2\right]|000\rangle$ from a partially reflecting mirror (say 99% reflection and 1% transmission) and adding through the mirror a field that has been phase and amplitude modulated according to the values μ, ν, τ, and β, γ, x. Thus the tripartite CES $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ can be implemented.

5. Some Applications of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$

In this section, we briefly introduce some probably applications of the tripartite CES.

5.1. Wigner Operator

In atomic and quantum optic area, Wigner distribution as the quasi-classical distribution [21, 22] well represent the non-classical properties of quantum state through its partial negativity in quadrature phase. One basic way to obtain Wigner distribution is to trace production between matrix density and Wigner operator [23]. Analogue to single mode Wigner function, and basing on the completeness and orthogonality of tripartite CES, we now introduce the following ket-bra integration

$$\int \frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{2\pi\sqrt{6\pi}}e^{3\mathrm{i}pu/2}\left|\beta,\gamma,x+\frac{u}{2}\right\rangle_{\mu\nu\tau\;\mu\nu\tau}\left\langle\beta,\gamma,x-\frac{u}{2}\right|\equiv\triangle(p,x) \tag{36}$$

Considering the explicit definition of $|\beta,\gamma,x\rangle_{\mu\nu\tau}$ in Equation (10) and employ the IWOP technique, we can directly calculate out

$$\begin{aligned}\triangle(p,x)=\frac{1}{\pi\tau^2\lambda^2}:\exp\Bigg[&-3\left(\frac{x}{\sqrt{2}}-\frac{\mu X_1+\nu X_2+\tau X_3}{3\lambda}\right)^2\\&-3\left(\frac{p}{\sqrt{2}}-\frac{\mu P_1+\nu P_2+\tau P_3}{3\lambda}\right)^2\Bigg]:\end{aligned} \tag{37}$$

which is a generalization of the normally ordered form of the usual Wigner operator. We may integrate $\Delta(p,x)$ out of x p, respectively, e.g.

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}x\Delta(p,x)&=\sqrt{\frac{2}{3\pi}}\frac{1}{\tau^2\lambda^2}:\exp\left[-3\left(\frac{1}{\sqrt{2}}p-\frac{\mu P_1+\nu P_2+\tau P_3}{3\lambda}\right)^2\right]: &(38a)\\&=\frac{1}{9}\sqrt{\frac{6}{\pi}}\int\frac{\mathrm{d}^2\sigma}{\pi}\frac{\mathrm{d}^2\kappa}{\pi}\left|\sigma,\kappa,p\right\rangle_{\mu\nu\tau\;\mu\nu\tau}\left\langle\sigma,\kappa,p\right| &(38b)\\\int_{-\infty}^{+\infty}\mathrm{d}p\Delta(p,x)&=\sqrt{\frac{2}{3\pi}}\frac{1}{\tau^2\lambda^2}:\exp\left[-3\left(\frac{1}{\sqrt{2}}x-\frac{\mu X_1+\nu X_2+\tau X_3}{3\lambda}\right)^2\right]: &(38c)\\&=\frac{1}{9}\sqrt{\frac{6}{\pi}}\int\frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}\left|\beta,\gamma,x\right\rangle_{\mu\nu\tau\;\mu\nu\tau}\left\langle\beta,\gamma,x\right| &(38d)\end{aligned}$$

Following Wigner's original idea of setting up a function in x-p phase whose marginal distribution is the probability of finding a particle in coordinate space and momentum space, respectively. we can immediately judge that the wigner operator $\Delta(p,x)$ in equations (38d) and (38b) is just a marginal distributional Wigner operator. Then the marginal distribution in the p-direction and its conjugate marginal distributions in the x-direction are

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{d}x\left\langle\psi\right|\Delta(p,x)\left|\psi\right\rangle\\&=\sqrt{\frac{2}{3\pi}}\frac{1}{\tau^2\lambda^2}\left\langle\psi\right|:\exp\left[-3\left(\frac{1}{\sqrt{2}}p-\frac{\mu P_1+\nu P_2+\tau P_3}{3\lambda}\right)^2\right]:\left|\psi\right\rangle\\&=\sqrt{\frac{1}{6\pi}}\int\frac{\mathrm{d}^2\sigma}{\pi}\frac{\mathrm{d}^2\kappa}{\pi}\left|\left\langle\psi|\sigma,\kappa,p\right\rangle_{\mu\nu\tau}\right|^2\end{aligned} \tag{39a}$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \mathrm{d}p \left\langle \psi \right| \Delta(p,x) \left| \psi \right\rangle \\
&= \sqrt{\frac{2}{3\pi}} \frac{1}{\tau^2\lambda^2} \left\langle \psi \right| : \exp\left[-3\left(\frac{1}{\sqrt{2}}x - \frac{\mu X_1 + \nu X_2 + \tau X_3}{3\lambda} \right)^2 \right] : \left| \psi \right\rangle \\
&= \sqrt{\frac{1}{6\pi}} \int \frac{\mathrm{d}^2\beta}{\pi} \frac{\mathrm{d}^2\gamma}{\pi} \left| \left\langle \psi | \beta, \gamma, x \right\rangle_{\mu\nu\tau} \right|^2 \qquad (39b)
\end{aligned}$$

correspondingly. Furthermore, we should note that in this case the classical x-p phase space corresponds to the operators X and P respectively.

5.2. Three-Mode Squeezing Operator

One important application of IWOP technique is to construct squeezed operator no matter how complex the quantum states is in continuous variable quadrature space [26,27]. In a similar way, We take a classical transformation $x \to \frac{x}{\eta}$ in $\left| \beta, \gamma, x \right\rangle_{\mu\nu\tau}$ to build a ket-bra integral,

$$S(\eta) = \tau^2\lambda^2 \int \frac{1}{\pi^2} \mathrm{d}^2\beta \mathrm{d}^2\gamma \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{6\eta\pi}} \left| \beta, \gamma, x/\eta \right\rangle_{\mu\nu\tau} {}_{\mu\nu\tau}\left\langle \beta, \gamma, x \right| \qquad (40)$$

Using the IWOP technique, we can directly perform the integration in Equation (40) to obtain

$$\begin{aligned}
S(\eta) = &\operatorname{sech}^{1/2}(\zeta) \exp\left\{ -\frac{1}{6\lambda^2} (\mu a_1^\dagger + \nu a_2^\dagger + \tau a_3^\dagger)^2 \tanh\zeta \right\} \\
&: \exp\left\{ \frac{1}{3\lambda^2} (\operatorname{sech}\zeta - 1)(\mu a_1^\dagger + \nu a_2^\dagger + \tau a_3^\dagger)(\mu a_1 + \nu a_2 + \tau a_3) \right\} : \\
&\exp\left\{ \frac{1}{6\lambda^2} (\mu a_1 + \nu a_2 + \tau a_3)^2 \tanh\zeta \right\} \qquad (41)
\end{aligned}$$

which is a new three-mode squeezing operator with parameter η, and where $\eta = \exp(\zeta)$, $\operatorname{sech}\zeta = 2\eta/(1+\eta^2)$ and $\tanh\zeta = (\eta^2-1)/(1+\eta^2)$. To make this squeezing more compact, we introduce the notation $R^\dagger = \frac{\mu a_1^\dagger + \nu a_2^\dagger + \tau a_3^\dagger}{\sqrt{3}\lambda}$, and using the following formula : $\exp[(e^\zeta - 1)a^\dagger a] := \exp(\zeta a^\dagger a)$, we can rewrite the Equation (41) as

$$\begin{aligned}
S(\eta) &= \operatorname{sech}^{1/2}(\zeta) \exp\left\{ -\frac{1}{2} R^{\dagger 2} \tanh\zeta \right\} \\
&\quad \times : \exp\left\{ (\operatorname{sech}\zeta - 1) R^\dagger R \right\} : \exp\left\{ \frac{1}{2} R^2 \tanh\zeta \right\} \\
&= \operatorname{sech}^{1/2}(\zeta) \exp\left\{ -\frac{1}{2} R^{\dagger 2} \tanh\zeta \right\} \\
&\quad \times \exp\left\{ R^\dagger R \ln\operatorname{sech}\zeta \right\} \exp\left\{ \frac{1}{2} R^2 \tanh\zeta \right\} \qquad (42)
\end{aligned}$$

And we can also find that R compose a $SU(1,1)$ Lie algebra as

$$[R, R^\dagger] = 1, \qquad [\frac{1}{2}R^2, \frac{1}{2}R^{\dagger 2}] = R^\dagger R + \frac{1}{2} \qquad (43)$$

The squeezing operator $S(\eta)$ squeezes states $\left| \beta, \gamma, x \right\rangle_{\mu\nu\tau}$ in a natural way

$$S(\eta) \left| \beta, \gamma, x \right\rangle_{\mu\nu\tau} = \frac{1}{\sqrt{\eta}} \left| \beta, \gamma, x/\eta \right\rangle_{\mu\nu\tau} \qquad (44)$$

Correspondingly, the three-mode squeezed vacuum state is

$$S(\eta)\left|000\right\rangle = \mathrm{sech}^{1/2}(\lambda)\exp\left\{-\frac{1}{6}(\mu a_1^\dagger + \nu a_2^\dagger + \tau a_3^\dagger)^2 \tanh\lambda\right\}\left|000\right\rangle \tag{45}$$

Using Equation (42) and the Baker-Hausdroff formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots \tag{46}$$

we see that

$$S(\eta)a_i S(\eta)^{-1} = a_i + \frac{\mu_i}{\sqrt{3}\lambda}\left[R(\cosh\zeta - 1) + R^\dagger \sinh\zeta\right] \tag{47}$$

where $\mu_{1,2,3} = \mu, \nu, \tau$. It then follows from Equation (8) that

$$S(\eta)X_i S(\eta)^{-1} = X_i + \mu_i A(e^{\zeta} - 1) \tag{48a}$$
$$S(\eta)P_i S(\eta)^{-1} = P_i + \mu_i B(e^{-\zeta} - 1) \tag{48b}$$

and $A = \sum_j \mu_j X_j/(3\lambda^2)$, $B = \sum_j \mu_j P_j/(3\lambda^2)$. So, under the $S(\eta)$ transformation the three quadrtures of the three-mode optical field become

$$S(\eta)(X_1 + X_2 + X_3)S(\eta)^{-1} = X_1 + X_2 + X_3 + (\mu + \nu + \tau)A(e^{\zeta} - 1) \tag{49a}$$
$$S(\eta)(P_1 + P_2 + P_3)S(\eta)^{-1} = P_1 + P_2 + P_3 + (\mu + \nu + \tau)B(e^{-\zeta} - 1) \tag{49b}$$

Operating $S(\eta)^{-1}$ on the three-mode vacuum state, we obtain the squeezed vacuum state

$$S(\eta)^{-1}\left|000\right\rangle = \mathrm{sech}^{1/2}\zeta \exp\left[\frac{\tanh\zeta}{2}R^{\dagger 2}\right]\left|000\right\rangle \equiv \left|\ \right\rangle_\rho \tag{50}$$

The expectation values of the two quadratures in this state are

$$_\rho\left\langle\ \right|(X_1 + X_2 + X_3)\left|\ \right\rangle_\rho = 0, \qquad _\rho\left\langle\ \right|(P_1 + P_2 + P_3)\left|\ \right\rangle_\rho = 0 \tag{51}$$

thus the variance of the two quadrature are

$$_\rho\left\langle\ \right|\Delta(X_1 + X_2 + X_3)^2\left|\ \right\rangle_\rho = {}_\rho\left\langle\ \right|(X_1 + X_2 + X_3)^2\left|\ \right\rangle_\rho \tag{52a}$$
$$= \frac{1}{2}\left[\frac{(\mu + \nu + \tau)^2}{3\lambda^2}(e^{2\zeta} - 1) + 3\right] \tag{52b}$$
$$_\rho\left\langle\ \right|\Delta(P_1 + P_2 + P_3)^2\left|\ \right\rangle_\rho = {}_\rho\left\langle\ \right|(P_1 + P_2 + P_3)^2\left|\ \right\rangle_\rho \tag{52c}$$
$$= \frac{1}{2}\left[\frac{(\mu + \nu + \tau)^2}{3\lambda^2}(e^{-2\zeta} - 1) + 3\right] \tag{52d}$$

and the minimum uncertainty relation is

$$\Delta(X_1 + X_2 + X_3)^2 \Delta(P_1 + P_2 + P_3)^2 = \frac{1}{4}\left[\frac{(\mu + \nu + \tau)^2}{3\lambda^2}(e^{2\zeta} - 1) + 3\right]\left[\frac{(\mu + \nu + \tau)^2}{3\lambda^2}(e^{-2\zeta} - 1) + 3\right] \tag{53}$$

In particular, when $\mu = \nu = \tau = 1$, the squeezed vacuum state $\left|\ \right\rangle_\rho$ reduces to the usual three-mode squeezed vacuum state, equations (52*b*), (52*d*),respectively, become

$$_\rho\left\langle\ \right|\Delta(X_1 + X_2 + X_3)^2\left|\ \right\rangle_\rho = \frac{3}{2}e^{2\zeta} \tag{54}$$

$$_\rho\left\langle\ \right|\Delta(P_1 + P_2 + P_3)^2\left|\ \right\rangle_\rho = \frac{3}{2}e^{-2\zeta} \tag{55}$$

$$\Delta(X_1 + X_2 + X_3)\Delta(P_1 + P_2 + P_3) = \frac{9}{4} \tag{56}$$

as expected. On the other hand, due to $3\lambda^2 \geq \mu\nu + \nu\tau + \mu\tau$, i.e. $(3\lambda)^2 \geq (\mu + \nu + \tau)^2$. For $\zeta \geq 0$, from equations $(52b)$and $(52d)$, we have

$$ {}_{\rho}\langle \, | \Delta(X_1 + X_2 + X_3)^2 | \, \rangle_{\rho} = \frac{1}{2}\left[\frac{(\mu+\nu+\tau)^2}{3\lambda^2}(e^{2\zeta}-1)+3\right] \tag{57a} $$

$$ \leq \frac{3}{2}e^{2\zeta} \tag{57b} $$

$$ {}_{\rho}\langle \, | \Delta(P_1 + P_2 + P_3)^2 | \, \rangle_{\rho} = \frac{1}{2}\left[\frac{(\mu+\nu+\tau)^2}{3\lambda^2}(e^{-2\zeta}-1)+3\right] \tag{57c} $$

$$ \geq \frac{3}{2}e^{-2\zeta} \tag{57d} $$

which implies that the squeezed vacuum state $| \, \rangle_{\rho}$ may exhibit stronger squeezing in one quadrature than that of the usual two-mode squeezed vacuum state while exhibiting weaker squeezing in another quadrature.

6. Conclusion

In summary, we have brought out the ways to construct tripartite CES just contrary the traditional method and check it correckness. We analyzed some major properties of the tripartite CES, i.e. the completeness relation and partly orthogonality. And a simple experimental protocol to produce tripartite CES was also proposed by using an asymmetric BS, which provides a new way to predict new tripartite squeezed operator.

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References

[1] Amico L, Fazio R and Osterloh A, 2008 *Rev Mod Phys* **80** 517
[2] Cerf N J,Leuchs G and Polzik E S, 2007 *Quantum Information with Continuous Varibables of Atoms and light*(Imperial College Press)
[3] Braunstein S L and Loock P V, 2005 *Rev. Mod. Phy* **77** 513
[4] Osborne T J and Nielsen M A,2002 *Phy. Rev. A* **66** 032110
[5] Fan H Y and Klauder J R, 1994 *Phy. Rev.A* **49** 704
[6] Fan H Y and Lu H L, 2004 *J. Phys. A: Math. Gen* **37** 10993
[7] Fan H Y and Liu S G, 2007 *Int.Jour. Mod. Phy.A* **22** 4481
[8] Fan H Y, and Lu H L and Fan Y, Ann. Phy **321** 480 (2006)
[9] Fan H Y, Zaidi H R and Klauder J R, 1987 *Phy. Rev. D* **35** 1831
[10] Fan H Y, Xu Z H, 1994 *Phy.Rev. A* **50** 2951
[11] Wünsche A, 1999 *J.Opt. B: Quantum Semiclass. Opt.* **1** R11
[12] Dupertuis M A and Stenholm S, 1988 *Phy. Rev. A* **37** 1226
[13] Vogel K and Risken H, 1989 *Phy. Rev. A* **40** 2847
[14] Fan H Y and Wang W Q, 2006 Commun. Theor. Phys **46** 975
[15] Hu L Y and Fan H Y, 2009 *Euro. Lett.* **85** 60001
[16] For a review, see Fan H Y, *J.Opt.B: Quantum Semiclass.Opt.* **5** R147
[17] Fan H Y, 1990 *Phys. Rev. A* **41** 1526
[18] Hu L Y and Fan H Y, 2007 *J.Phys. B: Mol. Opt. Phys* **40** 2099
[19] Leonhardt U and Paul H, 1995 *Prog. Quant. Electr,* **19** 89
[20] Leonhardt U, 1993 *Phys. Rev. A* **48** 3265-3276.
[21] Li S B, Liu J, Zou X B, and Guo G C, 2008 *J. Opt. Soc. Am. B* **25** 54
[22] Biswas A and Agarwal G S, 2007 *Phys. Rev. A* **75**,032104

[23] Scully M O and Zubairy M S, 1997 *Quantum Optics* (Cambridge University Press)
[24] Fan H Y and Cheng H L,2001 *Chin. Phys. Lett.* **18** 850
[25] Fan H Y, 2002 *Phys. Rev. A* **65** 064102
[26] Fan H Y and Yu G C, 2002 *Phys. Rev. A* **65** 03389
[27] Fan H Y and VanderLinde J, 1989 *Phys. Rev. A* **40** 4789